Beurling Integers: Part 1

Introduction

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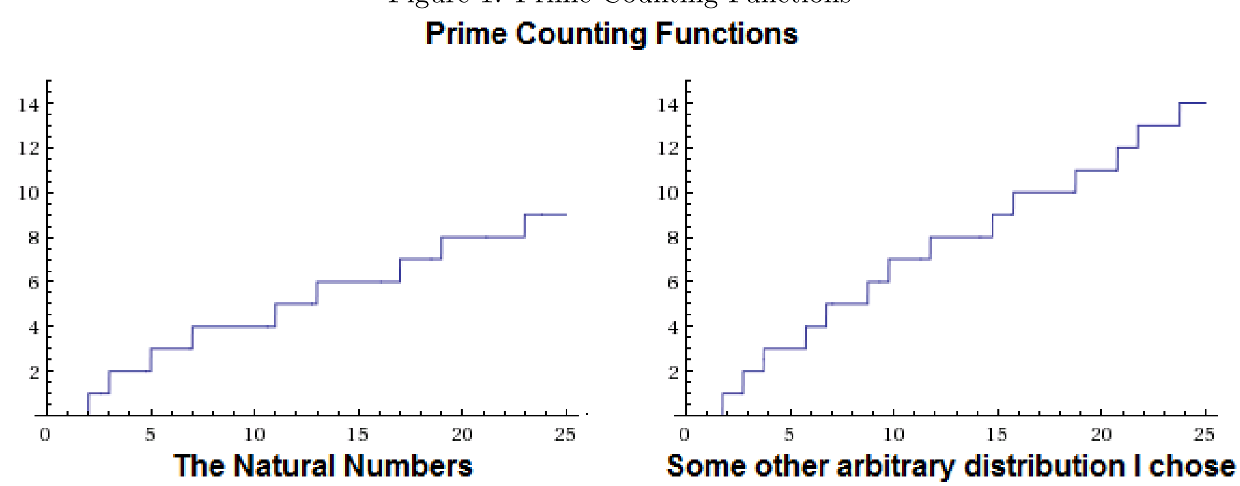
There is much about the distribution of the prime numbers that still remains a mystery. Key results like the Prime Number Theorem exist, but problems like the famous Riemann Hypothesis and the Twin Primes Conjecture (and many others) still remain unresolved. At its heart, the distribution of prime numbers is puzzling because although multiplication has a relatively simple and contained definition, the distribution of primes appears rather arbitrary. What is it about multiplication over the natural numbers that necessitates that, say, four is composite and five is prime? To answer the question, we might try to reverse engineer things: what would multiplication look like if we arbitrarily chose our own distribution of primes?

Figure 1: Prime Counting Functions



We can consider a multiplication as a function which maps two natural numbers to another natural number. A prime is identified as a number only mapped to by itself and an identity element, if we allow identity elements to exist (we would want this identity to be two-sided, so really we would at most have one identity element). By chaining multiplication, primes can act as a set of generators for all other numbers in the range. We want to preserve our ideas associated with primes, like uniqueness of prime factorization, also known as the fundamental theorem of arithmetic. By "uniqueness" we mean two things:

- Any set of prime numbers (with multiples of any prime allowed) is uniquely associated with a number as the product of the primes in that set.
- Any number is associated with only one such set of primes.

The first item is equivalent to having the commutative and associative properties over multiplication. However, the commutative and associative properties don't guarantee that there is only one prime factorization for any number. So when we refer to uniqueness of prime factorization we'll typically be referring the second item, as commutativity and associativity will be assumed. So suppose I wanted a system of integers where 5 wasnt prime. I could simply define my multiplication function m such that $m(2, 2) = 5$. Really I could choose any arguments besides an identity or 5. So for instance we could have $m(10001, 10^{39} - 1) = 5$ if we wanted. We actually have so many options that things are uninteresting. One important property integer multiplication that we might wish to impose in that it is increasing: $a, b < m(a, b)$ if neither a nor b are the identity. We are still left with one other troubling option. Suppose $a < b$. We would normally expect that for any c, $m(a, c) < m(b, c)$. The property that larger numbers lead to greater products is known as translation invariance, and it is the last restriction we impose outright on multiplication. Translation invariance leads to an extra consistency in multiplication. If we only impose increasingness, then situations like $m(2, 4) = 10$ and $m(2, 3) = 12$ are still possible, since $10 > 2, 4$ and $12 > 2, 3$. By outright restrictions I mean that any other restrictions we might wish to impose would be on the limiting behavior of multiplication, like requiring the infinitude of primes and infinitude of composites (as is the case of regular multiplication.) While limiting behavior is the focus of analysis, it actually isn't very relevant to the combinatorial problems I will be covering. (That said, the cases with a finite number of primes or composites are rather boring, and some proofs in part 2 regarding isomorphisms will require that we have an infinite number of both.) One consequence of increasingness and translation invariance is that if there is an identity element it will be the number 1. A simple proof can be given by contradiction. Suppose I is the identity element. Suppose that $a < I$. Then by translation invariance

$$m(a, a) < m(a, I) = a$$

Since a is not the identity element, this contradicts increasingness. We have now described a suitable generalization of multiplication that allows us to choose any distribution of primes and also maintains many important properties of integer multiplication. Whats noticeably left out is the distributive property of multiplication over addition. Without this property multiplication is completely unrelated to addition. However, if the distributive property is included with the previous properties then there is no generalization of multiplication; our distribution of primes will be completely defined. This is easy to prove provided that we have the multiplicative identity:

$$m(x, y) = m(x, 1 + 1 + ... + 1 \text{ (y times)}) = x + x + ...x \text{ (y times)} = x \cdot y$$

So to sum up, we can choose any distribution of primes we like, some set $P \subseteq N \setminus \{1\}$ (with $2 \in P$), and then associate it with a multiplication function $m : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following properties:

- Associative
- Commutative
- Uniqueness of Prime Factorization
- Identity Element (which is 1)
- Increasing
- Translation-invariant

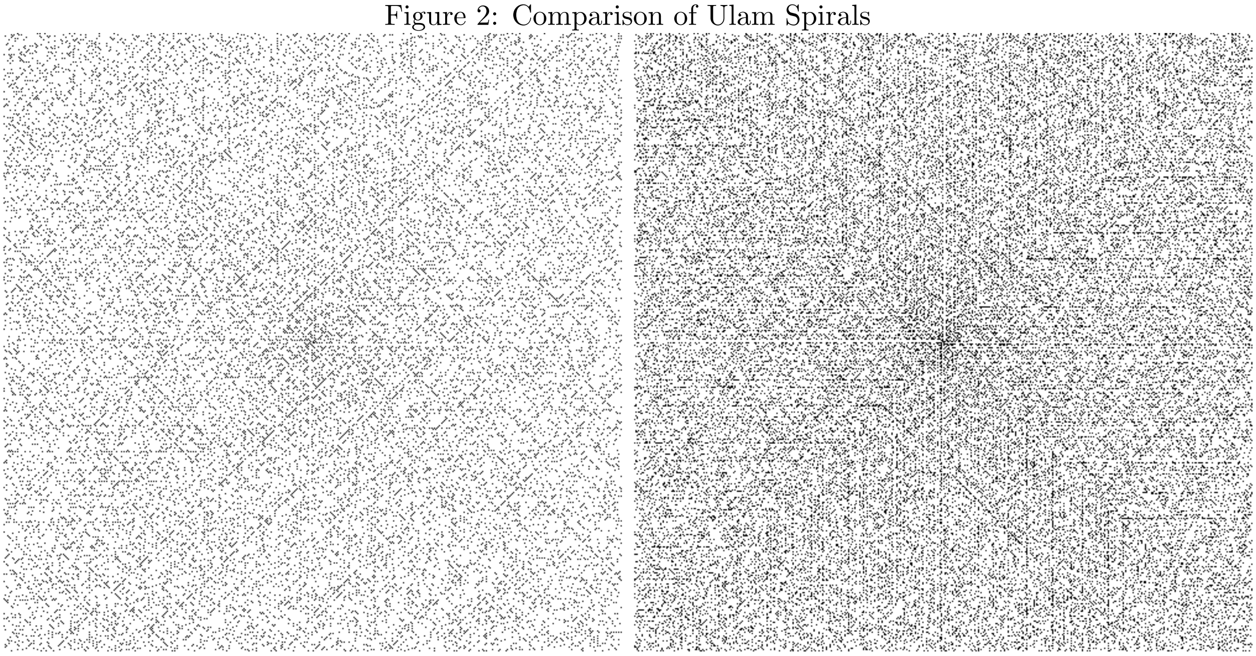
If $\mathbb{N} \setminus P$ is finite, then our number of composites is finite and m is only defined for a subset of $\mathbb{N} \times \mathbb{N}$. If P is of a finite size greater than one, then m is not unique. I am not sure whether m is unique when both P and $\mathbb{N} \setminus P$ are infinite. However, if we consider our problem as the distribution of primes and composites, then m is unique for our choice of ordering of all prime factorizations, since really these two things are one and the same. This isomorphism will be made clear in part 2, when we give a representation for "ordering of prime factorizations". At this point though, I think enough has been explained to warrant some examples. Example: The odd numbers The arbitrary distribution I gave earlier in this post was actually not arbitrary. It was that of the odd numbers, which we label here as odd(n):

n	1	2	3	4	5	6	7	8
odd(n)	1	3	5	7	9	11	13	15

As one can see, the distribution of primes for odd numbers clearly differs from the natural numbers in that the fourth number is prime and the fifth number is composite. In this case, the multiplication function has a value $m(2,2) = 5$, since the second number, 3, times itself equals the fifth number, 9. Here's a comparison of the multiplication tables for the natural numbers and the odd numbers.

Here's a comparison of their distributions of primes using the Ulam Spiral visualization (black represents a prime):

Figure 2: Comparison of Ulam Spirals



The Natural Numbers The Odd Numbers

The odd numbers make for a nice example because they have a closed form equation:

$$odd(n) = 2n - 1$$

We also have the inverse function: $odd^{-1}(n) = \frac{1}{2}(n+1)$ The definition of our multiplication function for the odds can be given in terms of this function and its inverse:

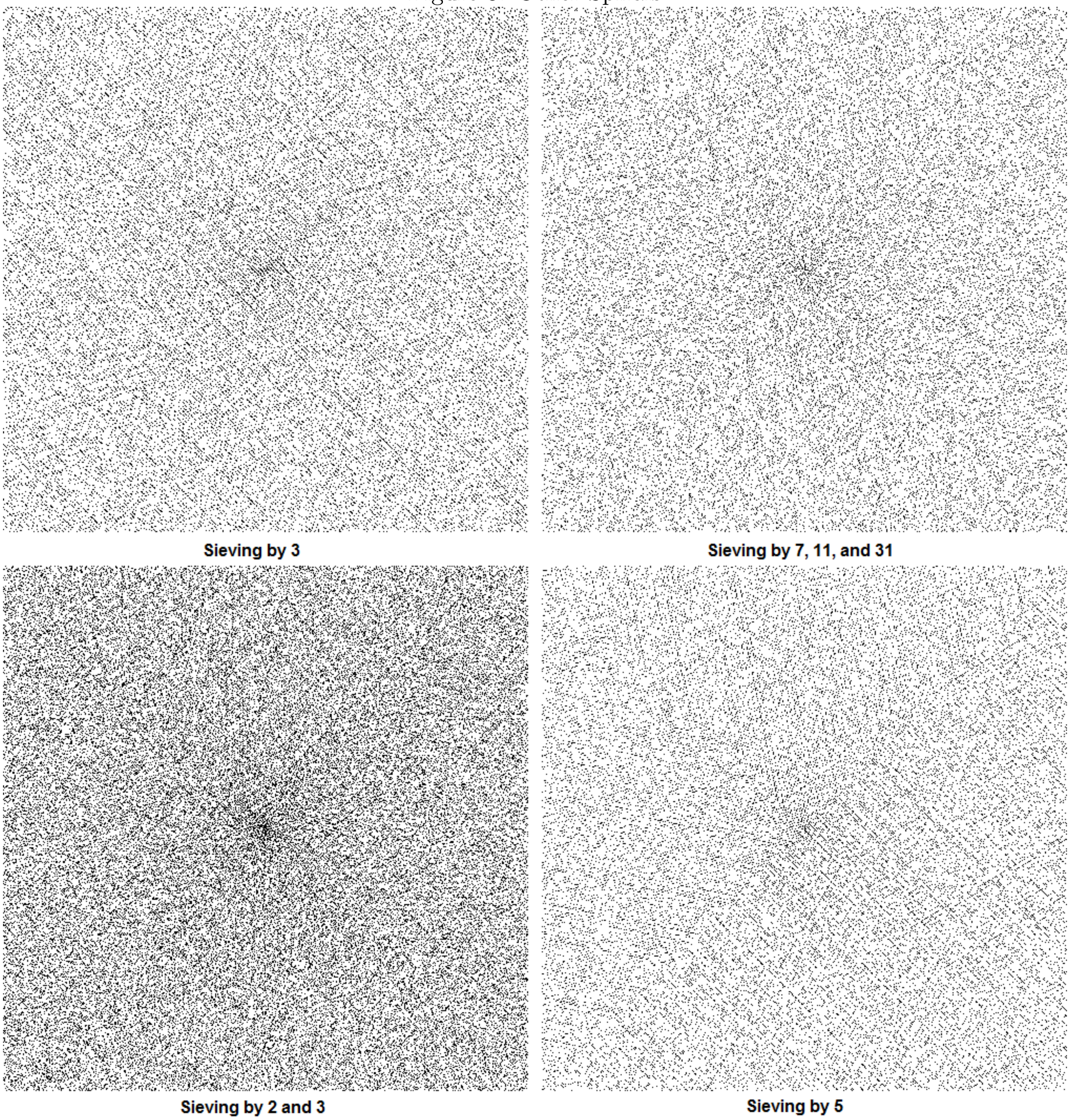
$$\begin{aligned} m(x,y) &= odd^{-1}(odd(x)\cdot odd(y)) \\ &= \frac{1}{2}((2x-1)\cdot(2y-1)+1) \\ &= \frac{1}{2}(4xy-2x-2y+1+1) \\ &= 2xy-x-y+1 \end{aligned}$$

What we're saying here is that we take the indices x and y, map to the odd numbers, multiply them using regular multiplication, and then map back to indices (using the inverse function). We can also give a formula of the odd number prime counting function in terms of the ordinary prime counting function:

$$\pi_{odd}(x) = \pi(2x-1) - 1$$

which holds for all $x > 1$. Taking note that 2 is prime and all other evens are composite, we can refine this to $\pi_{odd}(x) = \pi(2x) - 1$ for all x. Thus the odd numbers have roughly the distribution of $\pi(2x)$, and it follows from the prime number theorem that this is asymptotic to $2 \cdot pi(x)$. This distribution of primes is denser than that of the natural numbers (hence why the Ulam spiral picture is darker). There are lot's of other examples we can give, simply by sieving the natural numbers by a prime or primes. Here are the Ulam spirals for some other examples:

Figure 3: Other Spirals



These examples actually all have denser distributions than the natural numbers. That's because they're all subsequences of the natural numbers. Sequences of integers are sometimes easier to work with, but there's no reason we can't work with sequences of any type of real number. In general we might have some sequence a_n we use to define multiplication, which might not have a closed form equation. That's OK, as long as the sequence is bijective we can still define the multiplication function as:

$$m(x, y) = a_n^{-1}(a_x \cdot a_y)$$

However, for any general sequence this might not define multiplication consistent with the properties we require! At this point too, you might be wondering why we started defining multiplication in terms of these sequences anyways. There are two reasons. One, this allows us to completely and succinctly define another distribution of primes. Two, this is the way that the Beurling generalized primes (and integers) are defined in the mathematical literature. In his 1936 paper, “Analyse de la loi asymptotique de la distribution des nombres premiers généralisés”, Arne Beurling defined the generalized primes as:

$$1 < y_1 < y_2 < \ldots < y_n < \ldots$$

and the generalized integers as:

$$1 < x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$$

where the generalized integers are the products of these primes put in order. Because in our case we want a bijective sequence of numbers we'll make this definition slightly stricter. We require our primes to be such that the generated integers are a strictly monotonic sequence. In a sense we can consider the sequence y_n as a generating set for a sequence x_n which has multiplicities removed (which we will refer to unambiguously as a_n or or $a(n)$), and our sequence p_n of primes is the minimal generating set for any such a_n. Using real numbers allows us to define the exact relationship between two primes. For example, suppose my first prime is 2, and that I have the following requirement relating the first and second primes:

$$p_1^2 < p_2 < p_1^3$$

Then of course, $4 < p_2 < 8$. Now if the value of p_2 is chosen such that $p_2 < p_1^{2.5}$, we could say that $p_1^4 < p_2^2 < p_1^5$. If p_2 is chosen such that $p_2 > p_1^{2.5}$, we could say that $p_1^5 < p_2^2 < p_1^6$. The point is, from real values we can determine these inequalities, orderings of the powers of primes. As it turns out, the ordering of powers of primes has been proven isomorphic to the Beurling generalized integers, and thus is isomorphic orderings of all prime factorizations as well (see section four in this paper by Lapidus and Hilberdink). This again is the topic of the next blog post. I have yet to provide though proof that a sequence of real numbers as described yields a valid multiplication function, so I will end this post with such a proof: Since a_n is bijective we can define $m(x, y) = a_n^{-1}(a_x \cdot a_y)$. Since a_n is divergent and has infinitely many primes, we know that m is valid for all $x, y \in \mathbb{N}$. Commutativity is clear from the formula and commutativity of regular multiplication:

$$m(x, y) = a_n^{-1}(a_x \cdot a_y) = a_n^{-1}(a_y \cdot a_x) = m(y, x)$$

Associativity:

$$\begin{aligned} m(x, m(y, z)) &= a_n^{-1}(a_x \cdot a(a_n^{-1}(a_y \cdot a_z))) \\ &= a_n^{-}1(a_x \cdot a_y \cdot a_z) \\ &= a_n^{-1}(a(a_n^{-1}(a_x \cdot a_y)) \cdot a_z) = m(m(x, y), z) \end{aligned}$$

Uniqueness of Prime Factorization: this is present by design. We required that that p_n generated a_n with no extra multiplicities of elements. Identity Element (which is 1), assuming we define $a_1 = 1$: $m(1, x) = a_n^{-1}(a_1 \cdot a_x) = a_n^{-1}(a_x) = x$ Increasing: this follows from the fact that regular multiplication is increasing for numbers greater than 1, and that the sequence is increasing. Translation invariance: this follows from the fact that regular multiplication is translation-invariant, and that the sequence is increasing.