Periodicity in the Beurling Generalized Integers

(Incomplete Exploratory Work)

Devin Platt

July 12, 2015

In this article we examine periodicity of divisibility for Beurling generalized integers, in various forms.

Periodicity is of note in that it is the property which gives the primes their sort of statistical nature. This is sometimes poetically described as the "music" of the prime numbers. We note from the onset that uniqueness of prime factorization is required since otherwise we dont have a good definition for divisibility. This post will vary in its levels of mathematical formality as it covers a topic that I have only begun to study. I'll attempt to prove some statements, but unfortunately others will just have to be left as hunches because their proofs elude me. The natural numbers have periodicity of divisibility in the sense that every second number is even, every third number is divisible by three, every fourth number is divisible by four, and so on. Other sequences may exhibit weaker forms of periodicity.

If we take the natural numbers and sieve by a finite number of primes (including one that is not 2) we can observe such a weaker form of periodicity. For example, if we remove 2 and 3 as primes:

1	**5**	7	11	13	17	19	23	**25**	29	31	**35**	37	41	43	47	49	53	**55**	57

We dont have periodicity exactly, but due to the repeating +4,+2 pattern we have 2 out of any 10 consecutive integers are divisible by 5. The previous example is a subset of the natural numbers. Although not all sequences which are subsets of the natural numbers will have such repeating patterns, a sequence which contains any non-whole number certainly wont have any repeating patterns. We would still like to generalize periodicity to these cases. Consider taking the natural numbers and adding π as a prime:

1	2	3	π	4	5	6	2π	7	8	9	3π	π^2

We no longer have repeating patterns, but we would expect any integer to be divisible by π more often than 4 and less often than 3. In fact, on average we would expect N/π of any N consecutive integers to be divisible by π. This weaker form of periodicity is more nuanced and will be the focus of the rest of this post. We define the weaker forms of periodicity, then give a table summarizing the types of periodicity discussed:

- **Finite periodicity**: There exists an $N \in \mathbb{N}$ such that for each integer i, exactly i out of Ni consecutive integers are multiples of i.
- **Weak Periodicity**: Let $divides_n(m) = 1$ if the nth integer divides the mth, and be equal to zero otherwise. Let $S_n(m) = \sum_{k=1}^{m} divides_n(k)$. Then a system of Beurling generalized integers

has weak periodicity if for all $n \in \mathbb{N} \setminus \{1\}$ the limit $\lim_{m\to\infty} \frac{S_n(m)}{m}$ exists and

$$0 < \lim_{m\to\infty} \frac{S_n(m)}{m} < 1$$

Here is the table:

Label	Description	Defined By
P_0	Exact Periodicity	$\mathbb{N}$
P_f	Finite Periodicity	Finite Sieving
P_w	Weak Periodicity	$0 < \lim_{m\to\infty} S_n(m)/m < 1$

Note the hierarchy:

$$P_0 \implies P_f \implies P_w$$

Equivalent statements to P_w:

- $\lim_{m\to\infty} \frac{S_n(m)}{m} =$ average frequency of divisibility by the nth integer.
- $\lim_{m\to\infty} \frac{S_n(m)}{m} = 1/a_n$, where $\{a_n\}$ is the sequence of Beurling integers normalized so that $\lim_{m\to\infty} \frac{S_2(m)}{m} = 1/a_2$.

The first statement isn't entirely formal, but it also should be fairly obvious from the limit. The second statement follows (in concept) from the isomorphism of Beurling integers to certain orders on $\mathbb{N}^2$ (see this post), and will be proven later in this post. It turns out that this type of normalization coincides with normalization in the sense of linearization of sequences with polynomial growth. Weak periodicity is equivalent to the following two (rough) conditions:

- There cannot be "too many" primes.
- There must be "enough" primes.

To see this, note that if there are "too many" primes $\frac{S_n(m)}{m} \to 0$, while if there are "too few" $\frac{S_n(m)}{m} \to 1$. Thus these conditions are necessary for weak periodicity. That these conditions are sufficient will stem from our proof that certain bounds on the growth of the integer counting function $N(x)$ imply weak periodicity.

$$N(x) = \text{ the number of integers } \leq x$$

The growth of $N(x)$ is tied to the growth of the primes.

$$\pi(x) = \text{ the number of primes } \leq x$$

(Note that $\pi(n) \neq \pi(x)$. There are two different ways of generalizing the prime counting function: as a function over the reals and as one over the natural numbers. $\pi(n) =$ the number of primes $\leq$ the nth integer.) Unless otherwise noted, in this post a sequence a_n refers to our sequence of generalized integers (and $N(x)$ is the integer counting function associated with a_n).

Proposition 1: Weak periodicity is equivalent to the following statement:

$$0 < \lim_{x\to\infty} \frac{N(x)}{N(cx)} < 1, \quad \forall c > 1$$

Proposition 2: If a Beurling system has uniqueness of prime factorization, weak periodicity is equivalent to regular growth of the integer counting function $N(x)$:

$$\exists\alpha > 0 \text{ such that } \forall c > 0, \frac{N(cx)}{N(x)} \sim c^{\alpha}$$

Proposition 3: If a Beurling system has uniqueness of prime factorization, weak periodicity is equivalent to saying that the Beurling system can be represented by a sequence with linear asymptotic growth. By this we mean that there is a sequence a_n representing our system for which for all $y > 0$

$$\lim_{x\to\infty} \frac{N(yx)}{N(x)} = y$$

Furthermore, this normalized sequence a_n obeys the following statement:

$$a_n = \lim_{m\to\infty} \frac{m}{S_n(m)}$$

Before we prove the propositions we require some lemmas.

Lemma 1: $\lim_{m\to\infty} \frac{S_n(m)}{m}$ and $\lim_{x\to\infty} \frac{N(x)}{N(a_n x)}$ are equal for all n, where a_n is our sequence of Beurling integers.

Lemma 2: Suppose weak periodicity. Then for all $y > 0$

$$\lim_{x\to\infty} \frac{N(yx)}{N(x)} \text{ exists}$$

We note that for now Lemma 2 is unproven (though some justification for its assumption of truth is provided). In the future I will amend this blog post, hopefully with a proof of Lemma 2.

Lemma 3: If the integer at index h is the product of the integers at indices f and g, then we have:

$$\lim_{m\to\infty} \frac{S_h(m)}{m} = \left(\lim_{m\to\infty} \frac{S_f(m)}{m}\right) \cdot \left(\lim_{m\to\infty} \frac{S_g(m)}{m}\right)$$

From an intuitive standpoint this makes sense as it is analogous to multiplying independent probabilities.

Corollary 3: Let $z = wy$. Then

$$\lim_{x\to\infty} \frac{N(x)}{N(zx)} = \left(\lim_{x\to\infty} \frac{N(x)}{N(wx)}\right) \cdot \left(\lim_{x\to\infty} \frac{N(x)}{N(yx)}\right)$$

(Given that the limits on the right side exist.) Let $i(x,k)$ be the natural number power of p_1 which follows $p_1^{i(x,k)} \leq x^k < p_1^{i(x,k)+1}$. This function $i(x,k)$ exists for $x^k > p_1$, and thus for any $x > 1$ there exists a K such that $i(x,k)$ exists for all $k > K$. Note that for $x = p_n$ we have that $i(p_n, k) = f_n(k)$, where $f_n(k)$ is the function defined in the proof in my previous post proving the isomorphism of Beurling integers to certain orders on $\mathbb{N}^2$.

Lemma 4: Suppose that for all $n \in \mathbb{N}$ we have $0 < \lim_{m\to\infty} S_n(m)/m < 1$ (weak periodicity), then

$$\lim_{m\to\infty} \frac{m}{S_n(m)} = p_1^{\alpha}$$

where $\alpha = lim_{k\to\infty} \frac{i(a_n,k)}{k}$ and $p_1 = \lim_{m\to\infty} \frac{m}{S_2(m)}$.

Corollary 4.1: Suppose that we have weak periodicity. Let $y > 1$. Then $\lim_{x\to\infty} \frac{N(yx)}{N(x)} = p_1^{\alpha}$, where $\alpha = lim_{k\to\infty} \frac{i(y,k)}{k}$ and and $p_1 = \lim_{m\to\infty} \frac{m}{S_2(m)}$.

Corollary 4.2: Suppose that we have weak periodicity. Let $z < 1$. Then $\lim_{x\to\infty} \frac{N(zx)}{N(x)} = p_1^{-\alpha}$, where $\alpha = lim_{k\to\infty} \frac{i(1/z,k)}{k}$ and $p_1 = \lim_{m\to\infty} \frac{m}{S_2(m)}$. In Lemma 4 and its corollaries, note that $\alpha > 0$. That $\alpha \geq 0$ follows from the fact that $0 \leq \frac{i(a_n,k)}{k}$ for all n and k. We require that $0 < \lim_{m\to\infty} S_n(m)/m$ because otherwise $\lim_{m\to\infty} m/S_n(m)$ diverges. The fact that $\alpha \neq 0$ comes from that $\lim_{m\to\infty} S_n(m)/m < 1$.

Lemma 5: if a system of Beurling integers was weak periodicity, the sequence

$$a_n = \lim_{m\to\infty} \frac{m}{S_n(m)}$$

represents it.

Lemma 6: Suppose that we have weak periodicity. For all $y > 1$ we have $y = p_1^{\alpha}$, where $\alpha = lim_{k\to\infty} \frac{i(y,k)}{k}$ and $p_1 = \lim_{m\to\infty} \frac{m}{S_2(m)}$. For all $z < 1$ we have $z = p_1^{-\alpha}$, where $\alpha = lim_{k\to\infty} \frac{i(1/z,k)}{k}$ and $p_1 = \lim_{m\to\infty} \frac{m}{S_2(m)}$.

Corollary 6: Suppose that we have weak periodicity. Let $p_1 = \lim_{m\to\infty} \frac{m}{S_2(m)}$. Then for all $y > 0$

$$\lim_{x\to\infty} \frac{N(yx)}{N(x)} = y$$

Proof of Lemma 1

Let $b_m = a^{-1}(a_n a_m)$ be the subsequence of $\{1, 2, 3, 4, 5, ...\}$ that is the index of multiples of a_n. Then

$$S_n(b_m) = N(a_m) = m \text{ and } b_m = N(a_n a_m)$$

So

$$\frac{S_n(b_m)}{b_m} = \frac{N(a_m)}{N(a_n a_m)} \tag{1}$$

Suppose $\lim_{m\to\infty} S_n(m)/m = L$. Then $\lim_{m\to\infty} S_n(b_m)/b_m = L$ for any sequence b_m of natural numbers. Consider any x_m tending to infinity. For large enough m there exists an $i(m) \in \mathbb{N}$ such that

$$a_{i(m)} \leq x_m < a_{i(m)+1}$$

There exist $b_m = a_{i(m)}$, $c_m = a_{i(m)+1}$ (nondecreasing sequences real numbers) such that

$$N(b_m) = N(x_m) = N(c_m) - 1$$

and

$$N(a_n b_m) \leq N(a_n x_m) < N(a_n c_m)$$

so

$$\frac{N(c_m) - 1}{N(a_n c_m)} < \frac{N(x_m)}{N(a_n x_m)} \leq \frac{N(b_m)}{N(a_n b_m)}$$

We take the limit as $m \to \infty$, discarding the "-1" in $N(c_m) - 1$:

$$\lim_{m\to\infty} \frac{N(c_m)}{N(a_n c_m)} \leq \lim_{m\to\infty} \frac{N(x_m)}{N(a_n x_m)} \leq \lim_{m\to\infty} \frac{N(b_m)}{N(a_n b_m)}$$

Given our equality (1) discussed before we have:

$$\lim_{m\to\infty} \frac{S_n(a^{-1}(a_n c_m))}{a^{-1}(a_n c_m)} < \lim_{m\to\infty} \frac{N(x_m)}{N(a_n x_m)} \leq \lim_{m\to\infty} \frac{S_n(a^{-1}(a_n b_m))}{a^{-1}(a_n b_m)}$$

By supposing that $lim_{m\to\infty} S_n(m)/m = L$ we have that

$$L \leq \lim_{m\to\infty} \frac{N(x_m)}{N(a_n x_m)} \leq L$$

and thus $\lim_{m\to\infty} N(x_m)/N(a_n x_m) = L$. Now suppose $lim_{x\to\infty} N(x)/N(a_n x) = L$. Letting $x_m = a_m$, we get from our equivalence (1) that

$$\lim_{m\to\infty} \frac{S_n(b_m)}{b_m} = \lim_{m\to\infty} \frac{N(x_m)}{a_n x_m} = L$$

where $b_m = a^{-1}(a_n a_m)$ is the subsequence of $\{1, 2, 3, 4, 5, \dots\}$ that is the index of multiples of a_n. Since the limit exists, we have that

$$lim_{m\to\infty} \frac{S_n(m)}{m} = L$$

□

"Proof" of Lemma 2

Suppose $y > a_1$. It is easy to see that $\liminf_{x\to\infty} N(yx)/N(x)$ and $\limsup_{x\to\infty} N(yx)/N(x)$ exist since the values are bounded:

$$N(a_n x)/N(x) < N(yx)/N(x) < N(a_{n+1}x)/N(x)$$

for some n. Unfortunately, it appears difficult to show that the limit superior and limit inferior are equal. We can also show that $\lim_{x\to\infty} N(yx)/N(x)$ exists for all y of the form:

$$y = \frac{a_i}{a_j}$$

where $a_i, a_j \in \{a_n\}$. That is, for all y in the generalized rationals associated with our system. To prove this, suppose y and z are such that

$$\lim_{x\to\infty} N(yx)/N(x) \text{ and } \lim_{x\to\infty} N(zx)/N(x) \text{ exist}$$

Then

$$\frac{N(yx_n)}{N(zx_n)} = \frac{N(yx_n)/N(x)}{N(zx_n)/N(x)} = \frac{N(\frac{y}{z}y_n)}{N(y_n)}$$

using the substitution $x_n = \frac{1}{z}y_n$. Thus the limit

$$\lim_{n\to\infty} \frac{N(\frac{y}{z}y_n)}{N(y_n)} = \lim_{n\to\infty} \frac{N(yx_n)/N(x)}{N(zx_n)/N(x)} = \frac{\lim_{x\to\infty} N(yx)/N(x)}{\lim_{x\to\infty} N(zx)/N(x)}$$

If we could show that these generalized rationals are dense in the real numbers (like is the case with the regular rationals), we could get arbitrarily good bounds on $N(yx)/N(x)$ for any $y > 1$. The typical procedure to prove density requires a maximum bound on the gaps between integers, which we don't have a priori. Weak periodicity does put a bound on growth, but the bound is precisely what we are trying to prove in our propositions! Another way we could try to prove Lemma 2 is to imagine taking our Beurling system and creating a new one by adding y as a prime (assuming that y is not a generalized rational, which is fine since otherwise we know that the limit exists anyways.) It would seem reasonable that this new system also has weak periodicity, and that the corresponding values of

$$\lim_{x\to\infty}\frac{N(yx)}{N(x)}$$

go unchanged. It would then follow that the limit exists. At the moment though this argument remains as hand waving and I have yet to succeed making an actual proof by this method.

Proof of Lemma 3

It follows from a property of $S_n(x)$:

$$S_h(x) = S_f(S_g(x)) = S_g(S_f(x))$$

which exists due to translation invariance (the numbers divisible by some integer f, ie. its multiples $f \cdot \{a_n\} = f, fa_1, fa_2, \dots$, retain the order of a_n). Formally, if

$$\lim_{m\to\infty}\frac{S_f(m)}{m} = F, \text{ and } \lim_{m\to\infty}\frac{S_g(m)}{m} = G$$

then for any $\epsilon > 0$ there exist N_f, N_g such that for any $N'_f > N_f$ and for any $N'_g > N_g$ we have

$$\left|\frac{S_f(N'_f)}{N'_f} - F\right| < \epsilon, \text{ and } \left|\frac{S_g(N'_g)}{N'_g} - G\right| < \epsilon$$

Which implies that

$$N'_f(F-\epsilon) < S_f(N'_f) < N'_f(F+\epsilon), \text{ and } N'_g(G-\epsilon) < S_g(N'_g) < N'_g(G+\epsilon)$$

Combining the two inequalities we can get an inequality for $S_h(x)$. Let N' be such that $N' > N_g$ and $S_g(N') > N_f$. Then

$$(S_g(N'))(F-\epsilon) < S_f(S_g(N')) < (S_g(N'))(F+\epsilon)$$

so

$$N'(G-\epsilon)(F-\epsilon) < S_f(S_g(N')) < N'(G+\epsilon)(F+\epsilon)$$

and

$$N'(G-\epsilon)(F-\epsilon) < S_h(N') < N'(G+\epsilon)(F+\epsilon)$$

Thus for any $\epsilon > 0$, there exists an N_h such that for all $N' > N_h$

$$(G-\epsilon)(F-\epsilon) < \frac{S_h(N')}{N'} < (G+\epsilon)(F+\epsilon)$$

and we get that $\lim_{m\to\infty} S_h(m)/m = FG$. □

Proof of Corollary 3

Suppose that

$$\lim_{x\to\infty}\frac{N(x)}{N(wx)} = W \text{ and } \lim_{x\to\infty}\frac{N(x)}{N(yx)} = Y$$

By the definition of a limit, for all $\epsilon > 0$, there exists X_w, X_y such that for all X'_w and for all X'_y

$$\left|\frac{N(X'_w)}{N(wX'_w)}\right| < \epsilon \text{ and } \left|\frac{N(X'_y)}{N(yX'_y)}\right| < \epsilon$$

So

$$N(wX'_w)(W-\epsilon) < N(X'_w) < N(wX'_w)(W+\epsilon)$$

and

$$N(yX'_y)(Y-\epsilon) < N(X'_y) < N(yX'_y)(Y+\epsilon)$$

Let X'_y be large enough so that $yX'_y > X_w$ (WLOG). Then

$$N(wyX'_y)(W-\epsilon) < N(yX'_y) < N(wyX_y)(W+\epsilon)$$

and so

$$N(wyX'_y)(W-\epsilon)(Y-\epsilon) < N(X'_y) < N(wyX'_y)(W+\epsilon)(Y+\epsilon)$$

dividing through by $N(wyX'_y)$ we get

$$(W-\epsilon)(Y-\epsilon) < \frac{N(X'_y)}{N(wyX'_y)} < (W+\epsilon)(Y+\epsilon)$$

Thus for all $\epsilon > 0$ there exists an X such that for all $X' > X$

$$(W-\epsilon)(Y-\epsilon) < \frac{N(X')}{N(wyX')} < (W+\epsilon)(Y+\epsilon)$$

and so we have that $\lim x \to \infty \frac{N(X')}{N(wyX')} = WY$. $\square$

Proof of Lemma 4

For any n and any k, we have the inequality

$$p_1^{i(a_n,k)} < a_n^k < p_1^{i(a_n,k)+1}$$

This implies that

$$S_{a^{-1}(p_1^{i(a_n,k)})}(m) < S_{a^{-1}(a_n^k)}(m) < S_{i(a_n,k)+1}(m)$$

Dividing by m and taking limits, we get

$$\lim_{m\to\infty}\frac{m}{S_{a^{-1}(p_1^{i(a_n,k)+1})}(m)} \le \lim_{m\to\infty}\frac{m}{S_{a^{-1}(a_n^k)}(m)} \le \lim_{m\to\infty}\frac{m}{S_{a^{-1}(p_1^{i(a_n,k)})}(m)}$$

By the third lemma we can take out the powers in the limits

$$\left(\lim_{m\to\infty}\frac{m}{S_2(m)}\right)^{i(a_n,k)} \le \left(\lim_{m\to\infty}\frac{m}{S_n(m)}\right)^{k} \le \left(\lim_{m\to\infty}\frac{m}{S_2(m)}\right)^{i(a_n,k)+1}$$

Taking the kth root, we get

$$\left(\lim_{m\to\infty}\frac{m}{S_2(m)}\right)^{\frac{i(a_n,k)}{k}} \le \left(\lim_{m\to\infty}\frac{m}{S_n(m)}\right) \le \left(\lim_{m\to\infty}\frac{m}{S_2(m)}\right)^{\frac{i(a_n,k)+1}{k}}$$

We can let $p_1 = \lim_{m\to\infty}\frac{m}{S_2(m)}$ and let $\alpha = \lim_{k\to\infty} i(a_n,k)/k$. We then get that

$$p_1^\alpha \le \lim_{m\to\infty}\frac{m}{S_n(m)} \le p_1^\alpha$$

Thus we have

$$\lim_{m\to\infty}\frac{m}{S_n(m)} = p_1^\alpha$$

□

Proof of Corollary 4.1

the fact that $y > 1$ is important because it implies that there exists a K such that for all $k > K$, $i(y,k)$ exists. Thus the limit $\alpha = lim_{k\to\infty}\frac{i(y,k)}{k}$ exists. This proof is very similar to the previous proof of lemma 4, but it's worth running through in its entirety to make sure that we don't miss the places where it differs. For any $y > 1$ ($y \in \mathbb{R}$) and for any $k \in \mathbb{N}$ we have

$$p_1^{i(y,k)} < y^k < p_1^{i(y,k)+1}$$

This implies that

$$N(p_1^{i(y,k)}x) \le N(y^k x) \le N(p_1^{i(y,k)+1}x)$$

and so

$$\frac{N(p_1^{i(y,k)+1}x)}{N(x)} \le \frac{N(y^k x)}{N(x)} \le \frac{N(p_1^{i(y,k)}x)}{N(x)}$$

Taking limits, we get

$$\lim_{x\to\infty}\frac{N(p_1^{i(y,k)+1}x)}{N(x)} \le \lim_{x\to\infty}\frac{N(y^k x)}{N(x)} \le \lim_{x\to\infty}\frac{N(p_1^{i(y,k)}x)}{N(x)}$$

Using Corollary 3, we pull out the powers on the left and right sides (here we assume that the middle limit exists, which needs to be proven in Lemma 2. This is currently a hole in the proof!)

$$\left(\lim_{x\to\infty}\frac{N(p_1 x)}{N(x)}\right)^{i(y,k)+1} \le \left(\lim_{x\to\infty}\frac{N(yx)}{N(x)}\right)^k \le \left(\lim_{x\to\infty}\frac{N(p_1 x)}{N(x)}\right)^{i(y,k)}$$

Using Lemma 1, we have

$$\left(\lim_{m\to\infty}\frac{m}{S_2(m)}\right)^{i(y,k)+1} \le \left(\lim_{x\to\infty}\frac{N(yx)}{N(x)}\right)^k \le \left(\lim_{m\to\infty}\frac{m}{S_2(m)}\right)^{i(y,k)}$$

and letting $p_1 = \lim_{m\to\infty} m/S_2(m)$, we have

$$p_1^{\frac{i(y,k)+1}{k}} \le \lim_{x\to\infty}\frac{N(yx)}{N(x)} \le p_1^{\frac{i(y,k)}{k}}$$

so we have that

$$\lim_{x\to\infty}\frac{N(yx)}{N(x)} = p_1^\alpha$$

□

Proof of Corollary 4.2

Let $y = 1/z$. Then $y > 1$, and we can invoke Corollary 4. Let $y_m = x_m/z$, where x_m is any divergent sequence of real numbers.

$$\frac{1}{\lim_{x\to\infty} \frac{N(zx)}{N(x)}} = \lim_{x\to\infty} \frac{N(x)}{N(zx)} = \lim_{m\to\infty} \frac{N(y_m)}{N(zy_m)} = \lim_{m\to\infty} \frac{N(yx_m)}{N(x_m)} = p_1^{\alpha}$$

Thus, $\lim_{x\to\infty} \frac{N(zx)}{N(x)} = p_1^{-\alpha}$, where $\alpha = \lim_{k\to\infty} \frac{i(1/z,k)}{k}$. □

Proof of Lemma 5

Lemma 5 follows directly from Lemma 4, since the value α is the same value used in the proof of the isomorphism of orders on $\mathbb{N}^2$ to Beurling integers. Though stated as a lemma, this is a fundamental result on its own.

Proof of Lemma 6

Suppose that y ¿ 1. Remember that $\alpha = lim_{k\to\infty} i(y,k)/k$. By the definition of a limit, for any $\epsilon > 0$ there exists a K such for all $k > K$

$$\left| \frac{i(y,k)}{k} - \alpha \right| < \epsilon$$

so

$$\alpha - \epsilon < \frac{i(y,k)}{k} < \alpha + \epsilon$$

Since $p_1^{i(y,k)} < y^k < p_1^{i(y,k)+1}$, this implies that

$$p_1^{\alpha-\epsilon} < y < p_1^{\alpha+\epsilon}$$

Thus we have that $y = p_1^{\alpha}$. If $z < 1$, then just apply the argument above to $y = 1/z$ to get our result.

Proof of Corollary 6

This follows from Lemma 6 and corollaries 4.1 and 4.2.

Proof of Proposition 1

From Lemma 1 it is clear that the statement in Proposition 1 implies weak periodicity. In the other direction, from Lemma 6 and Corollary 6 we have that weak periodicity implies the statement in Proposition 1 because

$$\lim_{x\to\infty} \frac{N(yx)}{N(x)} = \left(\lim_{m\to\infty} \frac{m}{S_2(m)} \right)^{\lim_{k\to\infty} \frac{i(y,k)}{k}}$$

Remember that $\lim_{m\to\infty} \frac{m}{S_2(m)} > 1$ and $\lim_{k\to\infty} \frac{i(y,k)}{k} > 0$, so $\lim_{x\to\infty} \frac{N(yx)}{N(x)} > 1$.

Proof of Proposition 2

It is clear that the statement of regular growth implies the statement in Proposition 1, so regular growth implies weak periodicity. From corollaries 4.1 and 4.2, we can see that weak periodicity also implies regular growth.

Proof of Proposition 3

First we provide the proof from the direction assuming weak periodicity. Weak periodicity implies that the limit $\lim_{m\to\infty} S_n(m)/m$ exists. Thus any subsequence produces the same limit. Let a_n be our sequence of Beurling generalized integers with weak periodicity. Let b_n be any subsequence of a_n. Consider the subsequence of a_n formed by:

$$p_1 b_1, p_1 b_2, \dots p_1 b_k, \dots$$

Note that the number of integers divisible by p_1 less than or equal to $p_1 b_k$ is equal to the index of b_k:

$$S_2(a^{-1}(p_1 b_k)) = a^{-1}(b_k)$$

So if we take our limit using the indices of this subsequence we get

$$\lim_{k\to\infty} \frac{S_2(a^{-1}(p_1 b_k))}{a^{-1}(p_1 b_k)} = \lim_{k\to\infty} \frac{a^{-1}(b_k)}{a^{-1}(p_1 b_k)}$$

This limit is equivalent to

$$\lim_{k\to\infty} \frac{N(b_k)}{N(p_1 b_k)}$$

For any divergent sequence of real numbers x_n ($x_n > 1$) we can construct two corresponding sequences b_n and c_n which take values from a_n and obey the inequality:

$$\frac{N(b_k)-1}{N(p_1 b_k)} \leq \frac{N(x_k)}{N(p_1 x_k)} \leq \frac{N(c_k)}{N(p_1 c_k)}$$

The construction is as follows. For any large enough k there exists an m such that $a_m \leq x_k < a_{m+1}$. Thus $p_1 a_m \leq p_1 x_k < p_1 a_{m+1}$. We let $b_k = a_{m+1}$ be the closest integer just above x_k and we let $c_k = a_m$ be the closest integer just below or equal to x_k. (We note that this construction may lead to sequences b_n and c_n which are not strictly increasing and therefore not exactly subsequences of a_n, but strict monotonicity is not important here.) We have that $N(b_k) - 1 = N(x_k) = N(c_k)$, and $N(p_1 c_k) \leq N(p_1 p_k) \leq N(p_1 b_k)$, and the inequality follows. The fact that $N(x)$ diverges to infinity means that the -1 in the left side of the inequality can be ignored for the purpose of taking limits. From our assumption of weak periodicity, the left side and the right side of the inequality both have the same limit, and so then the middle does as well. Thus we get

$$0 < \lim_{x\to\infty} \frac{N(x)}{N(p_1 x)} < 1$$

Now in the other direction we have a similar method. For any sequence of real numbers x_n ($x_n > 1$) we have

$$\lim_{n\to\infty} \frac{N(x_n)}{N(p_1 x_n)} = \lim_{x\to\infty} \frac{N(x)}{N(p_1 x)}$$

In particular, this is true for $x_n = a_n$, our sequence of Beurling generalized integers. Thus we have

$$0 < \lim_{m\to\infty} \frac{S_2(m)}{m} < 1$$

To show that

$$0 < \lim_{m\to\infty} \frac{S_n(m)}{m} < 1$$

for any n, we recall the isomorphism of Beurling integers to an order on prime factorizations. From this we can take any integer a_n and give an inequality with powers of p_1, ie. for any n there is a j such that $p_1^j \leq a_n < p_1^{j+1}$. This implies that

$$\lim_{m\to\infty} \frac{S_q(m)}{m} \leq \lim_{m\to\infty} \frac{S_n(m)}{m} \leq \lim_{m\to\infty} \frac{S_r(m)}{m}$$

where q and r are the indices of these prime powers. By our lemma, these limits on the left and right sides of the inequality are strictly bounded between 0 and 1, since they are finite powers of the limit $\lim_{m\to\infty} S_2(m)/m$ on p_1. Finally, we have that for all $n > 1$

$$0 < \lim_{m\to\infty} \frac{S_n(m)}{m} < 1$$

□

Remarks

We end with some remarks on the relevance of periodicity. Top researchers in the field have commented that conditions on additive structure are probably necessary for Beurling integers to satisfy a generalized Riemann Hypothesis (see Matthew Watkins' discussion with Professor Harold Diamond about equal spacing: http://empslocal.ex.ac.uk/people/staff/mrwatkin//zeta/beurling.htm). Jeffrey Lagarias suggested the Delone property as one such condition. The Delone property states that there are positive constants r and R such that for all $n \in \mathbb{N}$

$$r \leq a_{n+1} - a_n \leq R$$

where a_n is our sequence of Beurling integers. The Delone property implies uniqueness of prime factorization since integers must all have multiplicity one.

In his paper (http://www.math.lsa.umich.edu/ lagarias/doc/beurling.pdf), Lagarias focuses on the case where a_n is a subsequence of the natural numbers. He proves that this case is exactly when a_n is generated by some finite change to the ordinary primes: taking some primes out, and possibly putting some new ones in (eg. 4 could become a prime if 2 was taken out.) We can see from this that Finite Periodicity implies the Delone property (moreover, the values r and R would be the minimum and maximum gaps in the periodic pattern of the system with finite periodicity). On the other hand, by its definition the Delone property implies that there exists a bound on the growth of a_n:

$$c_1 x < N(x) < c_2 x$$

for some constants c_1 and c_2. This is not quite enough to guarantee regular growth, but it is very close.

If a system of Beurling integers has uniqueness of prime factorization and satisfies $\lim_{n\to\infty} \frac{a_n}{n} = \alpha$ for some $\alpha > 0$ (ie. $a_n \sim \alpha n$), then it also has the Delone property since gaps between integers approach α as $n \to \infty$. I would like to think that the property in proposition 3

$$\lim_{x\to\infty} \frac{N(yx)}{N(x)} = y, \quad \forall y > 0$$

implies such a linearity condition, but I am not sure if this is true. If this were true, it would mean that when there is uniqueness of prime factorization, a property on the growth of $N(x)$ implies a condition on additive structure, which seems to go against the general intuition of researchers in the field. So there may be some error in my reasoning here.

In the discussion of Lemma 2 we mentioned the density of generalized rationals in the real numbers. It is clear that infinitude of integers (which implies a sort of Archimedean property) and a maximum bound on the gaps between integers imply such density (see here). It might be interesting to study the relationship between weak periodicity and density of generalized rationals, or just generalized rationals in their own right.

Although weak periodicity is not a condition on additive structure, it does have some ties to Delone property. It is also simply an important property from what we expect from the prime numbers anyways, while not being too restrictive. For example, weak periodicity opens up the Beurling numbers to probabilistic number theory.